ISSN:0971-1260
Vol-22-Issue-10-November-2019

Inference under progressive interval censoring for power function competing risk failure model.

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ABSTRACT

Competing risk failure model is used when failure of individuals or items may be due to more than one cause or factor. For such causes competing risk failure model is used instead of simple failure model. In this paper power function distribution is used as a life time model for competing risk failure analysis. The causes of failure are assumed independent. Under type-I interval censoring scheme the competing risk failure model is used for inference. The point estimation and interval estimation are done based on maximum likelihood as well as Bootstrap method. The estimation is carried out for the parameters and survival function of the competing risk failure method. Non parametric estimation of method is also used for estimation of survival function. A simulation study is done to study the behaviour of the estimator.

1. Introduction:

Failure data analysis is used in many fields like: engineering, medicine and many other fields. The failure time data arises as time to AIDS for HIV individuals, the time to death for cancer patients, life of electrical or electronic components etc.

The failure of an item may be due to one or more than one causes. When failure is observed due to multiple causes the analysis of such failure time data becomes more complicated. For example, Boag(1949) considered a data of a breast cancer patient where the cause of death is recorded as “ Cancer” or “other”. Death of patients may be due to Kidney dieses or by heart attack, such a competing risk failure model is studied by many authors like Peck(1966), Boardman & Kendell(1970), Hoel(1972), and Patel and Gajjar(1992).
Recently Gadhvi and Bhimani (2015, 2016) have considered competing risk failure model for geometric as well as exponential life time models.

In competing risk failure the causes of failure may be assumed independent or dependent. In most of the situations independent causes of failure is assumed for the data analysis, since there is some concern about the identifiability of the underlying model even though the assumption of dependent may be more realistic.

Suppose that an unit fails due to only one of the K different independent causes. When the unit begins operation each failure cause simultaneously generate a random life. Thus in effect, K lifetimes $X_1, X_2, \ldots X_K$ simultaneously begin, lifetime X_i corresponding to the i-th mode of failure. Failure of the unit occurs as soon as any one of the life times Xi, i = 1,2, ... ,K is realized. If X is a lifetime of the unit then we have,

$$X = \min\{X1, X_2, \ldots, X_K\} \equiv X_{(1)}$$

The cumulative distribution function(cdf) of X is given by,

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$$F_X(X) = 1 - \prod_{i=1}^{K} [1 - F_{Xi}(X)]$$

Here each failure mode can have any failure time distribution that not all failure distribution need be alike, but all the K modes operate independently of each other. Such a model is known as competing risk failure model.

May life time distributions like exponential, Weibull, Rayleigh, Pareto etc. are usually used as life time model. The power function distribution is a flexible life time distribution that may offer a good fit to some sets of failure data. Meniconi(1995) used the power function distribution to study reliability of the electrical component. He found that the power function distribution is the best compared to exponential, lognormal and Weibull life time models for reliability of the electrical component. The properties of power function distribution are studied by Johson and Kotz(1970).Many of the authors like Malik (1967). Ahsanullah (1974,1989), Munawar and Farooq(2012), Zarrin et al(2013), Zaka et al (2013) have considered two-parameter power function distribution. Zaka et al (2014) have used different methods of estimation of the parameters of power function distribution.

2. Competing Risk Failure model:

Suppose that the device or individual exhibits two models of failure and each failure mode simultaneously generate a random lifetime X_1 and X_2 respectively. i.e. X_i is the time of failure of an item due to cause i then the probability density function(pdf) of X_i, in case of power function distribution is given by,

$$f_i(x) = \theta_i x^{\theta_i - 1}, \ 0 < x < 1, \theta_i > 0, i = 1,2. \qquad ..(2.1)$$

and corresponding cumulative distribution function (cdf) is,

$$F_i(x) = x^{\theta_i}, i = 1, 2. \qquad ..(2.2)$$

Let T be the failure time of the item regardless of cause then T = min(X_1, X_2). The pdf of T is given by,

$$H_T(t) = P(T \le t)$$

$$= 1 - \prod_{i=1}^{2} \left[1 - F_{X_i}(t)\right]$$

$$= 1 - \left(1 - t^{\theta_1}\right)\left(\left(1 - t^{\theta_2}\right)\right)$$

$$= t^{\theta_1} + t^{\theta_2} - t^{\theta_1 + \theta_2} \qquad ..(2.3)$$

And its pdf can be obtained as,

$$h_T(t) = \frac{dH_T(t)}{dt}$$

$$= \theta_1 t^{\theta_1 - 1}(1 - t^{\theta_2}) + \theta_2 t^{\theta_2 - 1}(1 - t^{\theta_1}) \qquad ..(2.4)$$

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Let $P_i(t)$ be the probability that an item failed by cause $i$at the time and it must not failed by the cause $i' \neq i$ upto time t ; $i, i' = 1,2.$

Then $P_i(t) = f_i(t)[1 - F_{i'}(t)],\ i' \neq i = 1,2.$

Hence probability of failure of an item at time t is given by,

$$\begin{aligned} P(t) &= P_1(t) + P_2(t) \\ &= f_1(t)\left[1 - F_2(t)\right] + f_2(t)\left[1 - F_1(t)\right] \\ &= \theta_1 t^{\theta_1 - 1}(1 - t^{\theta_2}) + \theta_2 t^{\theta_2 - 1}(1 - t^{\theta_1}) \\ &= g_1(t) + g_2(t) \qquad \ldots (2.5) \end{aligned}$$

which is same as(2.4). Thus the model given by (2.5) is called power function competing risk failure model.

3. Interval Censoring:

Suppose that the failure observed between the given time interval (a, b), a < b. here experimental units. Here experimental units are not monitored continuously. In such situation interval censored data arise.

In medical studies such types of data arise when patients are only monitored at regular interval (weekly or quarterly check up). Many authors have discussed application of interval censoring in clinical, medical, bio-medical and engineering studies like Odell et al (1992), Sumuelson and Kongerud(1994), Scallan(1999), Rao(1998), Aggarwala(2001) etc. Such censoring scheme is also known as group censoring scheme. Patel and Gajjar(1992), Patel and Patel(2007) have considered maximum likelihood estimation under progressive grouped censored samples in case of compound exponential and compound geometric lifetime models with different parameters at different stages of censoring. Recently Gadhvi and Bhimani (2015, 2016) have considered geometric and exponential life time distributions under competing risk failure model keeping same parameter at all the stages of censoring.

Under progressive type I interval censored sample. The likelihood function is given by,

$$L(\theta_1, \theta_2) \propto \prod_{i=1}^{m} \left[G_1(T_i) - G_1(T_{i-1})\right]^{x_{1i}} \prod_{i=1}^{m} \left[G_2(T_i) - G_2(T_{i-1})\right]^{x_{2i}} \prod_{i=1}^{m} [1 - H(T_i)]^{r_i} \qquad ..(3.1)$$

Where r_i denotes the fixed numbering items removed at the end of the i-th time interval (T_{i-1} , T_i), i.e. at time T_i , i=1,2,….,m. here m = the fixed pre-determined number of time intervals; T_0=0 , n= total number of items put on a test, i.e. sample size.

Let x_{ji} denotes the number of failure during the time interval [T_{i-1} , T_i], i=1,2,…,m. due to j-th cause of failure, j=1,2.,

and

$$r_m = n - \sum_{i=1}^{m} x_{1i} - \sum_{i=1}^{m} x_{2i} - \sum_{i=1}^{m-1} r_i$$

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Here, $$G_1(T_i) = P(X_1 \le T_i)$$

$$= \int_0^{T_i} g_1(t)dt = \int_0^{T_i} \theta_1 t^{\theta_1 - 1}(1 - t^{\theta_2})dt$$

$$= T_i^{\theta_1} - \left(\frac{\theta_1}{\theta_1 + \theta_2}\right) T_i^{\theta_1 + \theta_2} \qquad ..(3.2)$$

Similarly, $$G_2(T_i) = P(X_2 \le T_i)$$

$$= \int_0^{T_i} g_2(t)dt = \int_0^{T_i} \theta_2 t^{\theta_2 - 1}(1 - t^{\theta_1})dt$$

$$= T_i^{\theta_2} - \left(\frac{\theta_2}{\theta_1 + \theta_2}\right) T_i^{\theta_1 + \theta_2} \qquad ..(3.3)$$

Using(2.3),(3.2) and (3.3) in (3.1) we get the likelihood function under progressive type-I interval censoring with two independent causes of failures for poser function lifetime model as,

$$L = L(\theta_1, \theta_2) = cons\tan t \prod_{i=1}^{m} \left\{ \left(T_i^{\theta_1} - T_{i-1}^{\theta_1}\right) - \left(\frac{\theta_1}{\theta_1 + \theta_2}\right)\left(T_i^{\theta_1 + \theta_2} - T_{i-1}^{\theta_1 + \theta_2}\right) \right\}^{x_{1i}} \times$$

$$\prod_{i=1}^{m} \left\{ \left(T_i^{\theta_2} - T_{i-1}^{\theta_2}\right) - \left(\frac{\theta 2}{\theta_1 + \theta_2}\right)\left(T_i^{\theta_1 + \theta_2} - T_{i-1}^{\theta_1 + \theta_2}\right) \right\}^{x_{2i}} \times \qquad ..(3.4)$$

$$\prod_{i=1}^{m} \left[(1 - T_i^{\theta_1})(1 - T_i^{\theta_2}) \right]^{r_i}$$

Taking log on both the sides, and after some algebraic manipulation, we have

$$LogL = -\sum_{i=1}^{m} x_{1i} \log(\theta_1 + \theta_2) + \sum_{i=1}^{m} x_{1i} \log\left\{ (\theta_1 + \theta_2)\left(T_i^{\theta_1} - T_{i-1}^{\theta_1}\right) - \theta_1\left(T_i^{\theta_1 + \theta_2} - T_{i-1}^{\theta_1 + \theta_2}\right) \right\} -$$

$$\sum_{i=1}^{m} x_{2i} \log(\theta_1 + \theta_2) + \sum_{i=1}^{m} x_{2i} \log\left\{ (\theta_1 + \theta_2)\left(T_i^{\theta_1} - T_{i-1}^{\theta_1}\right) - \theta_2\left(T_i^{\theta_1 + \theta_2} - T_{i-1}^{\theta_1 + \theta_2}\right) \right\} +$$

$$\sum_{i=1}^{m} r_i \log\left(1 - T_i^{\theta_1}\right) + \sum_{i=1}^{m} r_i \log\left(1 - T_i^{\theta_2}\right)$$

..(3.5)

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Let us define

$$\psi_{1i} = T_i^{\theta_1} - T_{i-1}^{\theta_1}, \quad \psi_{2i} = T_i^{\theta_2} - T_{i-1}^{\theta_2}, \quad \psi_{12i} = T_i^{\theta_1+\theta_2} - T_{i-1}^{\theta_1+\theta_2}$$

$$d\psi_{1i} = \frac{d\psi_{1i}}{d\theta_1}, \; d\psi_{2i} = \frac{d\psi_{2i}}{d\theta_2}, \quad d\psi_{12i} = \frac{d\psi_{12i}}{d\theta_1} = \frac{d\psi_{12i}}{d\theta_2}$$

$$d2\psi_{1i} = \frac{d^2\psi_{1i}}{d\theta_1^2}, \; d2\psi_{2i} = \frac{d^2\psi_{2i}}{d\theta_2^2}, \quad d2\psi_{12i} = \frac{d^2\psi_{12i}}{d\theta_1^2} = \frac{d^2\psi_{12i}}{d\theta_2^2}$$

Then

Differentiating logL with respect to θ_1 and θ_2 and comparing them with zero, we get

$$\frac{\partial \log L}{\partial \theta_1} = \frac{-\sum_{i=1}^{m} x_i}{\theta_1+\theta_2} + \sum_{i=1}^{m} x_{1i}\left\{\frac{\psi_{1i} + (\theta_1+\theta_2)d\psi_{1i} - \psi_{12i} - \theta_1 d\psi_{12i}}{(\theta_1+\theta_2)\psi_{1i} - \theta_1\psi_{12i}}\right\} + \qquad ..(3.6)$$

$$\sum_{i=1}^{m} x_{2i}\left\{\frac{\psi_{2i} - \theta_2 d\psi_{12i}}{(\theta_1+\theta_2)\psi_{2i} - \theta_2\psi_{12i}}\right\} + \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{r_i}\left(\frac{-T_i^{\theta_1}\log T_i}{1-T_i^{\theta_1}}\right) = 0$$

and

$$\frac{\partial \log L}{\partial \theta_2} = \frac{-\sum_{i=1}^{m} x_i}{\theta_1+\theta_2} + \sum_{i=1}^{m} x_{2i}\left\{\frac{\psi_{2i} + (\theta_1+\theta_2)d\psi_{2i} - \psi_{12i} - \theta_2 d\psi_{12i}}{(\theta_1+\theta_2)\psi_{2i} - \theta_2\psi_{12i}}\right\} + \qquad ..(3.7)$$

$$\sum_{i=1}^{m} x_{1i}\left\{\frac{\psi_{1i} - \theta_1 d\psi_{12i}}{(\theta_1+\theta_2)\psi_{1i} - \theta_1\psi_{12i}}\right\} + \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{r_i}\left(\frac{-T_i^{\theta_2}\log T_i}{1-T_i^{\theta_2}}\right) = 0$$

Solving (3.6) & (3.7) by iterative procedure we can get $\hat{\theta}_1$, mle of θ_1 and $\hat{\theta}_2$, mle of θ_2.
The survival function at time t_0 is given by
$S = s(t_0) = 1\text{-}H(t_0) = (1 - t^{\theta_1})(1 - t^{\theta_2})$
Replacing θ_1 and θ_2 by their MLEs we get MLE of survival function $\hat{S}(t_0)$ as

$$\hat{S}(t_0) = (1 - t^{\hat{\theta}_1})(1 - t^{\hat{\theta}_2}) \qquad ..(3.8)$$

4. Standard errors of the estimators:

The asymptotic variance and covariance matrix Σ of the MLE for the parameters θ_1 and θ_2 are given by elements of the inverse of the Fisher information matrix,

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$$I_{ij} = E\left(-\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}\right); \qquad i, j = 1, 2$$

That is,

$$\Sigma = \begin{pmatrix} V(\hat{\theta}_1) & Cov(\hat{\theta}_1, \hat{\theta}_2) \\ Cov(\hat{\theta}_1, \hat{\theta}_2) & V(\hat{\theta}_2) \end{pmatrix} = \begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial\theta_1^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial\theta_1 \partial\theta_2}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial\theta_1 \partial\theta_2}\right) & -E\left(\frac{\partial^2 \log L}{\partial\theta_2^2}\right) \end{bmatrix}^{-1} \quad ..(4.1)$$

where,

$$\frac{\partial^2 \log L}{\partial\theta_1^2} = \frac{\sum_{i=1}^{m} x_i}{(\theta_1+\theta_2)^2} - \sum_{i=1}^{m}\left\{\frac{r_i (\log T_i)^2 \, T_i^{\theta_1}}{(1-T_i^{\theta_1})^2}\right\} +$$

$$\sum_{i=1}^{m} x_{1i}\left[\frac{((\theta_1+\theta_2)\psi_{1i} - \theta_1\psi_{12i})\{2(d\psi_{1i} - d\psi_{12i}) + (\theta_1+\theta_2)d2\psi_{1i} - \theta_1 d2\psi_{12i}\} - \{\psi_{1i} - \psi_{12i} + (\theta_1+\theta_2)d\psi_{1i} - \theta_1 d2\psi_{12i}\}^2}{((\theta_1+\theta_2)\psi_{1i} - \theta_1\psi_{12i})^2}\right] +$$

$$\sum_{i=1}^{m} x_{2i}\left[\frac{((\theta_1+\theta_2)\psi_{2i} - \theta_2\psi_{12i})(-\theta_2 d2\psi_{12i}) - (\psi_{2i} - \theta_2 d\psi_{12i})^2}{((\theta_1+\theta_2)\psi_{2i} - \theta_2\psi_{12i})^2}\right]$$

..(4.2)

$$\frac{\partial^2 \log L}{\partial\theta_2^2} = \frac{\sum_{i=1}^{m} x_i}{(\theta_1+\theta_2)^2} - \sum_{i=1}^{m}\left\{\frac{r_i (\log T_i)^2 \, T_i^{\theta_2}}{(1-T_i^{\theta_2})^2}\right\} +$$

$$\sum_{i=1}^{m} x_{2i}\left[\frac{((\theta_1+\theta_2)\psi_{2i} - \theta_2\psi_{12i})\{2(d\psi_{2i} - d\psi_{12i}) + (\theta_1+\theta_2)d2\psi_{2i} - \theta_2 d2\psi_{12i}\} - \{\psi_{2i} - \psi_{12i} + (\theta_1+\theta_2)d\psi_{2i} - \theta_2 d2\psi_{12i}\}^2}{((\theta_1+\theta_2)\psi_{2i} - \theta_2\psi_{12i})^2}\right] +$$

$$\sum_{i=1}^{m} x_{1i}\left[\frac{((\theta_1+\theta_2)\psi_{1i} - \theta_1\psi_{12i})(-\theta_1 d2\psi_{12i}) - (\psi_{1i} - \theta_1 d\psi_{12i})^2}{((\theta_1+\theta_2)\psi_{1i} - \theta_1\psi_{12i})^2}\right]$$

..(4.3)

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$$\frac{\partial^2 \log L}{\partial\theta_1\partial\theta_2}=\frac{\sum_{i=1}^{m} x_i}{(\theta_1+\theta_2)^2}+\sum_{i=1}^{m} x_{1i}\left[\frac{\begin{array}{c}\{(\theta_1+\theta_2)\psi_{1i}-\theta_1\psi_{12i}\}\{d\psi_{1i}-d\psi_{12i}-\theta_1 d2\psi_{12i}\}-\\ (\psi_{1i}-\theta_1 d\psi_{12i})\{\psi_{1i}-\psi_{12i}+(\theta_1+\theta_2)d\psi_{1i}-\theta_1 d2\psi_{12i}\}\end{array}}{((\theta_1+\theta_2)\psi_{1i}-\theta_1\psi_{12i})^2}\right]+$$

$$\sum_{i=1}^{m} x_{2i}\left[\frac{\begin{array}{c}\{(\theta_1+\theta_2)\psi_{2i}-\theta_2\psi_{12i}\}\{d\psi_{2i}-d\psi_{12i}-\theta_2 d2\psi_{12i}\}-\\ (\psi_{2i}-\theta_2 d\psi_{12i})\{\psi_{2i}-\psi_{12i}+(\theta_1+\theta_2)d\psi_{2i}-\theta_2 d2\psi_{12i}\}\end{array}}{((\theta_1+\theta_2)\psi_{2i}-\theta_2\psi_{12i})^2}\right]$$

..(4.4)

The asymptotic variance of MLE of survival function is given by,

$$V\left(\hat{S}(t_0)\right)=\left(\frac{\partial S}{\partial\theta_1}\right)^2 V\left(\hat{\theta}_1\right)+\left(\frac{\partial S}{\partial\theta_2}\right)^2 V\left(\hat{\theta}_2\right)+2\left(\frac{\partial S}{\partial\theta_1}\right)\left(\frac{\partial S}{\partial\theta_2}\right)\operatorname{cov}\left(\hat{\theta}_1,\hat{\theta}_2\right) \qquad \ldots(4.5)$$

where, $\frac{\partial S}{\partial\theta_1}=(1-t^{\theta_2})(-t^{\theta_1}\log t)$ and $\frac{\partial S}{\partial\theta_2}=(1-t^{\theta_1})(-t^{\theta_2}\log t)$

5. Simulation Algorithm:

A study of properties of maximum likelihood estimators based on progressively Type-I interval censored samples involves simulation. A short algorithm for simulating a random sample of size n put on a life test at time 0 is given below. Here we use the following properties of the progressive interval censoring.

$X_{11}\sim B\left(n,G_1(T_1)\right)$ and $X_{21}\sim B\left(n,G_2(T_1)\right)$

Where $G_1(T_1)$ and $G_2(T_1)$ are defined in (3.3) and (3.4) respectively and for i = 2,3,…,m

$$X_{ji}/X_{ji-1},X_{ji-2},X_{ji-3,\ldots,}X_{j1},R_{i-1},R_{i-2},\ldots,R_1\sim B\left(n-\sum_{s=1}^{i-1}(X_{1s}+X_{2s}+R_s);\frac{G_j(T_i)-G_j(T_{i-1})}{1-\sum_{s=1}^{i-1}\{G_j(T_i)-G_j(T_{i-1})\}}\right)$$

$$=B\left(n-\sum_{s=1}^{i-1}(X_{1s}+X_{2s}+R_s);\frac{G_j(T_i)-G_j(T_{i-1})}{1-G_j(T_{i-1})}\right);\ j=1,2.$$

Here $B(n,p)$ denotes the binomial distribution with parameters $n$and p; $0<p<1$.
On the basis of the algorithm given in Aggarwala (2001) for simulating a sample under progressive Type-I interval censoring scheme Gadhvi and Bhimani (2015) have modify the simulation scheme for competing risk failure model. The main steps of the algorithm are given below:

1. Set $i=0, X_{1sum}=0, X_{2sum}=0, R_{sum}=0$

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2. Next i
3. If $i = m + 1$, exist the algorithm.
4. Generate X_{11} and X_{21} as binomial random variables with parameters $\left(n, G_1(T_1)\right)$ and $\left(n, G_2(T_1)\right)$ respectively.
5. Generate X_{1i} and X_{2i} as binomial variables with parameter
$\left(n - X_{1sum} - X_{2sum} - R_{sum}, \frac{G_1(T_i) - G_1(T_{i-1})}{1 - G_1(T_{i-1})}\right)$ and

$\left(n - X_{1sum} - X_{2sum} - R_{sum}, \frac{G_2(T_i) - G_2(T_{i-1})}{1 - G_2(T_{i-1})}\right)$ respectively.
6. Calculate $R_i^{obs} = Floor\left[p_i\left(n - X_{1sum} - X_{2sum} - R_{sum}\right)\right]$ or
$\min\left(R_i, n - X_{1sum} - X_{2sum} - R_{sum} - X_i\right)$.
7. Set $X_{1sum} = X_{1sum} + X_{1i}, X_{2sum} = X_{2sum} + X_{2i}, R_{sum} = R_{sum} + R_i^{obs}$
This algorithm generates m binomial random variables. Here either the values $p_1, p_2, \dots, p_{m-1}$ or proposed values of $R_1, R_2, \dots, R_m$ are fixed in advance by the experimenter. Here $p_m = 1$and $R_m = n - \sum_1^m X_{1i} - \sum_1^m X_{2i} - \sum_1^{m-1} R_i$.

6. Confidence Interval Estimation:

A. Asymptotic Confidence Interval:

Using the asymptotic normality property of maximum likelihood estimator, confidence interval for MLE can be obtained for parameters θ_1, $\theta_2$as and survival function $S(t_0)$ as follows:

$(1-\propto)100\%$ asymptotic confidence interval for $\hat{\theta}_i$ becomes

$$\hat{\theta}_i \pm Z_{\alpha/2}\sqrt{V\left(\hat{\theta}_i\right)}, i = 1, 2$$

And for $S(t_0)$:

$$\hat{S}(t_0) \pm Z_{\alpha/2}\sqrt{V\left(\hat{S}(t_0)\right)}$$

Where $V(\hat{\theta}_i)$ and $V\left(\hat{S}(t_0)\right)$ can be obtained from equations (3.14) & (3.18) and $Z\alpha/_2$ is$\left(\alpha/_2\right)^{th}$ percentile of standard normal distribution.

B. Bootstrap Confidence Interval:

B-I: Percentile Bootstrap Method:

1. From the original data X compute the ML estimates of the parameters say $\hat{\theta}_1$ and $\hat{\theta}_2$ from (3.12) and (3.10).
2. Use $\hat{\theta}_1$ and $\hat{\theta}_2$ to generate a bootstrap sample $\underline{X}^*$with the same values of T_i, r_i and m, $i = 1, 2, \dots, m$, using the algorithm given in section 5.
3. As discussed in step 1, based on $\underline{X}^*$ compute the bootstrap sample estimator of θ_1, θ_2 and $S(t_0)$, say $\hat{\theta}_1^*, \hat{\theta}_2^*$ and $\hat{S}^*(t_0)$.
4. Repeat steps 2 and 3, S times representing S bootstrap MLE's of $\left(\theta_1, \theta_2, S(t_0)\right)$ based on S different bootstrap samples.

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5. Arrange all $\hat{\theta}_1^*, \hat{\theta}_2^*$ and $\hat{S}^*(t_0)$ in an ascending order to obtain bootstrap sample $\left(\phi_l^{[1]}, \phi_l^{[2]}, ..., \phi_l^{[S]}\right), l = 1, 2, 3.$ where $\phi_1 = \hat{\theta}_1^*, \phi_2 = \hat{\theta}_2^*$ and $\phi_3 = \hat{S}^*(t_0)$.
6. Let $G(Z) = P(\phi_l \le Z)$ be the cumulative distribution function of ϕ_l. Define $\phi_{lboot} = G^{-1}(Z)$ for given Z. The approximate bootstrap $(1-\alpha)100\%$ confidence interval of ϕ_l is given by $\left[\phi_{lboot}\left(\alpha/2\right), \phi_{lboot}\left(1-\alpha/2\right)\right]$.

B-II: Bootstrap-t method:

1. From the original data $\underline{x}$ compute the MLE of the parameters say, $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{S}(t_0)$.
2. Use $\hat{\theta}_1 and \hat{\theta}_2$ to generate bootstrap sample $\underline{X}^*$ with the same values of T_i, r_i and m, $i = 1, 2, \dots, m$, using the algorithm given in B-I.
3. Based on $\underline{X}^*$ compute the bootstrap sample estimates of θ_1, θ_2 and $S(t_0)$, say $\hat{\theta}_1^*, \hat{\theta}_2^*$ and $\hat{S}^*(t_0)$.
4. Compute the following statistics:

$$T_1^* = \frac{\sqrt{n}\left(\hat{\theta}_1^* - \hat{\theta}_1\right)}{\sqrt{V\left(\hat{\theta}_1^*\right)}}, T_2^* = \frac{\sqrt{n}\left(\hat{\theta}_2^* - \hat{\theta}_2\right)}{\sqrt{V\left(\hat{\theta}_2^*\right)}}, T_3^* = \frac{\sqrt{n}\left(\hat{S}^*(t_0) - \hat{S}(t_0)\right)}{\sqrt{V\left(\hat{S}^*(t_0)\right)}}.$$

   where $V(\,)$ can be obtained from (3.14).
5. Repeat step 3 and 4 S boot times.
6. From the values of $T_i^*, i = 1,2,3$ obtained in step 4, determine the upper and lower bounds of the $100(1-\alpha)\%$ confidence interval of the parameters and survival function as follows:
   Let $H(x) = P\left(T_i^* \le x\right), i = 1, 2, 3$ be the cdf of T_i^*. For a given x define;
   $\left(\hat{\theta}_{1Boot-t}\left(\alpha/2\right), \hat{\theta}_{1Boot-t}\left(1-\alpha/2\right)\right)$. Similarly we can define other parameters.

7. Non Parametric Estimation of Survival Function.

Kaplan Meier estimate of survival function $S(t_0)$ can be obtained according to Miller et al (1981 pg. 47,51). The estimate and estimate of the variance can be obtained as follow:

$$\hat{S}(t_0) = \prod_{Y_{(i)} \le t_0} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}} \qquad ..(7.1)$$

and

$$Asy\hat{V}\left(\hat{S}(t_0)\right) = \hat{S}^2(t_0) \sum_{y_{(i)} \le t_0} \frac{\delta_{(i)}}{(n-i)(n-i+1)} \qquad ..(7.2)$$

Using the results (7.1) & (7.2), the $(1-\propto)100\%$ asymptotic confidence interval for $\hat{S}(t_0)$ can be obtained

$$\hat{S}(t_0) \pm Z_{\alpha/2} \sqrt{Asy\hat{V}\left(\hat{S}(t_0)\right)} \qquad ..(7.3)$$

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8. Simulation study:

In this section we have simulated the data considering the parametric values θ_1 = 2.2, θ_2 = 2.4 and time t = 0.6 for reliability R(t). A sample of size 350 is generated for 10 stages of censoring with two causes of failures using the using the algorithm given in Section 5.

Number i	**Time interval (T_{i-1}, T_i)**	**Type – I Failures: x_{1i}**	**Type – II Failures: x_{2i}**	**Withdrawals R_i**
1	0-0.1	3	1	5
2	0.1-0.2	8	4	5
3	0.2-0.3	11	13	4
4	0.3-0.4	10	18	4
5	0.4-0.5	26	21	3
6	0.5-0.6	15	18	3
7	0.6-0.7	25	22	3
8	0.7-0.8	14	21	2
9	0.8-0.9	13	4	1
10	0.9-1.0	7	2	64

As per our notations we have $S_i = T_i - T_{i-1} = 0.1$, m =10, T_m = 1.0, T_0 = 0, n = 350, r_m = 64. Solving the equations (3.8) to (3.10) of section 3 we find MLEs of θ_1 and θ_2 as.

$$\widehat{\theta_1} = 2.65547976116334 \quad and \quad \widehat{\theta_2} = 2.77379446782239 \qquad ..(8.1)$$

From (3.12) we get MLE for survival function at time t_0 = 0.6 as

$$\hat{S}(t_0 = 0.6) = 0.562426155196962$$

Using (4.1) the asymptotic variance –covariance matrix of the MLE s for parameter θ_1 and θ_2 is given by

$$\begin{pmatrix} V(\widehat{\theta_1}) & Cov(\widehat{\theta_1}, \widehat{\theta_2}) \\ Cov(\widehat{\theta_1}, \widehat{\theta_2}) & V(\widehat{\theta_2}) \end{pmatrix} = \begin{pmatrix} 0.0239166 & 0.0001783 \\ 0.0001783 & 0.0429117 \end{pmatrix}$$

Hence the asymptotic standard errors of the MLEs of the parameters will be

$$SE(\widehat{\theta_1}) = 0.154650081 \qquad and \qquad SE(\widehat{\theta_2}) = 0.207151404$$

Also from (4.5) we get

$$V(\hat{S}(t_0 = 0.6)) = 0.00060369$$

and its standard error

$$SE(\hat{S}(t_0 = 150)) = 0.024570262$$

To apply bootstrap confidence interval estimation for the parameters we have made 1000 simulations based on the MLEs of θ_1 and θ_2 given in (8.1) and fixing the other values

$S_i = T_i - T_{i-1} = 0.1$, m =10, T_m = 1.0, T_0 = 0, n = 350 .

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The summary statistics for our simulation are given in the following table:

Table:1 Summary statistics for simulation

	min	0.025	0.25	0.50	0.75	0.975	max
$\widehat{\theta_1}$	0	2.6752114	2.9625226	3.1127389	3.2639031	3.5846541	4.0248624
$\widehat{\theta_2}$	0	2.8363811	3.1353680	3.3030604	3.4663995	3.8030967	4.1674482
$\hat{S}(t_0)$	0	0.5892184	0.6286557	0.6476742	0.6677814	0.7039539	0.7411426
$T\widehat{\theta_1}$	-327.9135 585	-4.644906 1E-06	-3.781503 4E-06	-2.784598 2E-06	3.772613 7E-06	4.582161 2E-06	5.306542 8E-06
$T\widehat{\theta_2}$	-157.7800 12779787	5.6960599 7269612	28.852003 1269008	39.017157 6332494	47.977305 9514092	63.077342 694876	74.291362 2974984
$T\hat{S}(t_0)$	-Infinity	20.423658 1956493	50.600252 7868964	64.719203 8331239	80.636558 2930453	111.49685 4184635	140.18979 491929

Based on the simulated results the confidence interval based on MLE and bootstrap confidence intervals for parameters and survival function are computed using the methods described in the Section 6, which are given in the following table.

Table:2 Estimates and confidence intervals for the parameters based on MLE and Boot strap

Method	Parameter	Estimate	Confidence Interval (Length of the Interval)
MLE	θ_1	3.1033260	(2.7468865, 3.4597655) (0.712879)
	θ_2	3.2874563	(2.7873602, 3.7875524) (1.0001922)
	$S(t_0 = 0.6)$	0.6437608	(0.5961844, 0.6913372) (0.0951528)
Percentile Bootstrap	θ_1	3.1033260	(1.9671556, 2.6358935) (0.66873789)
	θ_2	3.2874563	(2.0230712, 2.7125887) (0.6895175)
	$S(t_0 = 0.6)$	0.6437608	(0.4493521, 0.5368521) (0.0875)
Bootstrap - t	θ_1	3.1033260	(2.6554797, 2.6554798) (0.0000001)
	θ_2	3.2874563	(2.8368652, 3.4722314) (0.6353662)
	$S(t_0 = 0.6)$	0.6437608	(0.5892493, 0.7088592) (0.1196099)

The following tables gives estimate of survival (Reliability) function ($S(t_0)$) of an item at time t_0, Its asymptotic variance and asymptotic 95% confidence interval of $S(t_0)$ based on MLE and nonparametric estimation as discussed in the sections 6 and 7.

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Table 3. Estimate of survival (Reliability) function ($S(t_0)$), Its asymptotic variance and asymptotic confidence interval using MLE:

t_0	$\hat{S}(t_0)$	AsyV($\hat{S}(t_0)$)	95% Confidence interval for $\hat{S}(t_0)$: Lower limit	Upper limit	Length of CI
0.1	0.9915886	2.5890868E-07	0.9906494	0.9925278	0.0018784
0.2	0.9809212	8.7065397E-06	0.9752926	0.9865498	0.0112572
0.3	0.9503237	5.9201947E-05	0.9354542	0.9651933	0.0297391
0.4	0.8890148	0.0001973	0.8616955	0.9163341	0.0546386
0.5	0.7881406	0.0004149	0.7483993	0.8278819	0.0794826
0.6	0.6437608	0.0005934	0.5961844	0.6913372	0.0951528
0.7	0.4610417	0.0005610	0.4148019	0.5072816	0.0924797
0.8	0.2598564	0.0002947	0.2263904	0.2933224	0.066932
0.9	0.0819732	4.4753505E-05	0.0689651	0.0949813	0.0260162

Table 4. Estimate of survival (Reliability) function ($S(t_0)$), Its asymptotic variance and asymptotic confidence interval using nonparametric estimation.

t_0	$\hat{S}(t_0)$	Asy V($\hat{S}(t_0)$)	95% Confidence interval for $\hat{S}(t_0)$ Lower limit	Upper limit	Length of CI
0.1	0.9961288	1.097749E-05	0.9961073	0.9961503	0.0000430
0.2	0.9750960	5.951185E-05	0.9547414	0.9862366	0.0314952
0.3	0.9265835	0.00018899	0.8947459	0.9492654	0.0545195
0.4	0.8453192	0.00037746	0.8033544	0.8796314	0.0762770
0.5	0.7339187	0.00057605	0.6843331	0.7782212	0.0938881
0.6	0.5998204	0.00071902	0.5463081	0.6510458	0.1047377
0.7	0.4644135	0.00075562	0.4111562	0.5184992	0.1073430
0.8	0.3450574	0.00069741	0.2952954	0.3984813	0.1031859
0.9	0.2624664	0.00060576	0.2171754	0.3134528	0.0962774

Here we see that from time point 0.2 and more than that asymptotic variance of the estimate of survival function and length of confidence interval based on MLE are smaller than that of based on non parametric estimation.

Acknowledgements

The authors are grateful to the anonymous referees and the Editor for their time and patience to review the paper.

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