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Relation Between The Roots Of Polynomial And Term Of Recurrence Relation Sequence

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Abstract
A recurrence relation is important topic of mathematics. Recurrence relations are used in mathematics as well as economics; physics and others subjects. Recurresive techniques are very useful for calculation. In this paper we present important theorem on relation between initial terms of sequence and roots polynomial. We discuss some example of that theorem.

Keywords
Sequence, recurrence relation, number theory, coefficient.

Introduction
In Number Theory there are many special types of Sequences Fibonacci Sequence and Luca Sequence both are special type of recurrence relation with given initial terms. Recurrence relation is very useful topic of mathematics. It is an equation that defines a sequence based on a method that gives the next term as relation of the previous terms [1, 2, 3]. Recurrence relation is very useful in mathematics as well as economics. We can calculate growth in economics by recurrence techniques. In recurrence relation for finding any term of sequence we need to find all previous terms of sequence but by using this theorem we can find direct any term of sequence. Recurrence relation is very useful in real life problems.

Preliminaries
Many papers have contributed to method of solving Recurrence relations such as [4, 5 and 6]. We classify recurrence relations the number of previous terms needed to find the new term.

First Order Recurrence Relation
In the first order recurrence relation only one initial term is given. For example
$a_{n+1} = a_n + 5, n \geq 1, a_0 = 0$

we can find the terms

$a_1 = 6, a_2 = 7, a_3 = 8$

Second Order Recurrence Relation
In the second order recurrence relation new term depend on two previous terms and two initial terms are given.

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For example

$$a_n = a_{n-1} + 2a_{n-2}, n \geq 2$$

with the initial terms $a_0 = 0, a_1 = 1$

Third Order Recurrence Relation

In the 3rd order recurrence relation new term is depend on previous three terms. For example

$$a_n = a_{n-3} + 2a_{n-2} + a_{n-1}, n \geq 3$$

with the initials terms $a_0 = 0, a_1 = 1, a_2 = 2$.

Main theorem of paper

Theorem: - Let c_1, $c_2\, and\, c_3$ are arbitrary real numbers and suppose the equation

$$x^3 - c_1x^2 - c_2x - c_3 = 0 \quad (1)$$

has $x_1, x_2\, and\, x_3$ are distinct roots. Then the sequence $< a_n >$ is a solution of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} \;\; n \geq 3 \quad (2)$$

iff

$$a_n = \beta_1x_1^n + \beta_2x_2^n + \beta_3x_3^n$$

for n= 0, 1, 2, ……………… where $\beta_1, \beta_2\, and\, \beta_3$ are arbitrary constants.

Proof: - First suppose that $< a_n >$ of type $a_n = \beta_1x_1^n + \beta_2x_2^n + \beta_3x_3^n$ we shall prove $< a_n >$ is a solution of recurrence relation (2). Since $x_1, x_2\, and\, x_3$ are roots of equation (1) so all are satisfied equation (1) so we have

$$x_1^3 = c_1x_1^2 + c_2x_1 + c_3$$

$$x_2^3 = c_1x_2^2 + c_2x_2 + c_3$$

$$x_3^3 = c_1x_3^2 + c_2x_3 + c_3$$

Consider $c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} = c_1(\beta_1x_1^{n-1} + \beta_2x_2^{n-1} + \beta_3x_3^{n-1}) + c_2(\beta_1x_1^{n-2} + \beta_2x_2^{n-2} + \beta_3x_3^{n-2}) + c_3(\beta_1x_1^{n-3} + \beta_2x_2^{n-3} + \beta_3x_3^{n-3})$

$=\beta_1x_1^{n-3}(c_1x_1^2 + c_2x_1 + c_3) + \beta_2x_2^{n-3}(c_1x_2^2 + c_2x_2 + c_3) + \beta_3x_3^{n-3}(c_1x_3^2 + c_2x_3 + c_3)$

$=\beta_1x_1^n + \beta_2x_2^n + \beta_3x_3^n = a_n$

This implies

$c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} = a_n$

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So the sequence $< a_n >$ is a solution of the recurrence relation.

Now we will prove the second part of theorem

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$ $n \geq 3$ is a sequence with three $initial terms$ $a_0 = A_1, a_1 = A_2, a_2 = A_3$

Let $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n$

So $\beta_1 + \beta_2 + \beta_3 = A_1$ (3)

$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = A_2$ (4)

$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 = A_3$ (5)

The system of linear equations has a non-trivial solution iff $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} \neq 0$

We know that $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ (6)

equation (6) is always non-zero as roots are distinct. So non- trivial values of β_1, β_2 and β_3 can find and we can say that result is valid.

Example

Let $< a_n >$ be any sequence such that $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $n \geq 3$ and $a_0 = 0, a_1 = 1, a_2 = 2$. Then find a_{10} for above sequence.

Solution: The polynomial of above sequence is $x^3 - 6x^2 + 11x - 6 = 0$. Solving this equation we have roots are 1, 2, and 3 using above theorem we have

$a_n = \beta_1 1^2 + \beta_2 2^2 + \beta_3 3^2$ (7)

Using $a_0 = 0, a_1 = 1, a_2 = 2$ in (7) we have

$\beta_1 + \beta_2 + \beta_3 = 0$ (8)

$\beta_1 + 2\beta_2 + 3\beta_3 = 1$ (9)

$\beta_1 + 4\beta_2 + 9\beta_3 = 2$ (10)

Solving (8), (9) and (10) we have $\beta_1 = -\frac{3}{2}, \beta_2 = 2, \beta_3 = -\frac{1}{2}$

this implies $a_n = -\frac{3}{2}1^n + 22^n - \frac{1}{2}3^n$

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now put n=10 we have $a_{10} = -27478$

Conclusion

Recurrence relation is very useful topic of mathematics many problems of real life many be solved by recurrence relations but in recurrence relation there is a major difficulty in the recurrence relation if we want find 100th term of sequence then we need to find all previous 99 terms of given sequence then we can get 100th term of sequence but above theorem is very useful if coefficients of recurrence relation of given sequence are satisfied the condition of the above theorem then we can apply above theorem and we can find direct any term of sequence without find all previous terms.

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