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Computation of Displacement In Horizontal and Vertical Oscillator An Application of Laplace Transform

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Abstract

Presently in current scenario, Mathematics has extended itself in pretty much every field of applied science, designing and research issues. Science has given different devices to taking care of their issues. One of the valuable numerical apparatuses is Laplace Transform and Inverse Laplace Transform. Laplace Transform has various applications in understanding the frameworks involving Differential Equations . Few most frequently encountered applications are in analysis of electric circuit, mechanical system, signal processing, heat transfer. This chapter deals with problems of vibrations of object attached to spring and flow of current in LCR circuit using Laplace transform.

Introduction

Incredible number of issues exists in mechanical and electrical framework which includes differential condition and incomplete differential condition. Now and then it gets tricky to comprehend differential conditions legitimately. Laplace change is a convenient apparatus as it changes over the framework into mathematical conditions which are similarly simple to deal with. Laplace Transform is utilized in numerical demonstrating. Because of the way that Laplace Transform is straightforward and apply, it is widely favored as appropriate numerical instrument to settle differential conditions and integro-differential conditions. Laplace Transform was named after the Mathematician Pierre-Simon Laplace who presented it while creating likelihood hypothesis. Later it was created by Oliver Heaviside in the structure it is being utilized today. Laplace change, meant by L_z , chips away at genuine variable τ, for the most part signifying time area, and returns a component of complex variable z.[1]

Let $g(\tau)$ be function of $\tau > 0$. Laplace Transform is defined as

$$L_z\big(g(\tau)\big) = \int_0^\infty g(\tau)e^{-z\tau}\, d\tau = G(z), \qquad \text{provided it exists}$$

$$L_z^{-1}\big(G(z)\big) = g(\tau), \text{ we say that inverse Laplace Transform of } G(z) \text{ is } g(\tau).$$

Applications of Laplace Transform

In this section, few elementary applications have been solved using Laplace Transform. Few unsolved problems of [2] have been considered.

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Displacement in Horizontal or Vertical Oscillator

Illustration1: An object of weight 0.25N is attached to coiled spring, where it is initially at rest. Find spring constant, natural frequency and Time period of oscillation when spring is extended by 0.06m. [2]

Solution: Force=$mg = 0.25$, $m = \frac{0.25}{9.8} = 0.026$

Extension in String $=$ 0.06 m

Considering the magnitude, spring constant = $\frac{F}{x} = \frac{0.25}{0.06} = 4.167$N/m

Natural frequency $= w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.167}{0.026}} = \sqrt{160.27}$ rad/sec

Period of oscillation= $\frac{2\pi}{w_0} = 0.496 \cong 0.5$ sec

Illustration2: The upper end of spring in above problem is given vertical displacement of $Z(\tau)$, with zero initial displacement. Let $X(\tau)$ be the displacement of object on the lower end of coiled spring. Find the differential equation that will be satisfied by $X(\tau)$ (ignoring damping forces). The vertical displacement of object from the equilibrium position when attached to spring while $Z(\tau) = 0$ is denoted as $Y(\tau)$. Show that differential equation of $Y(\tau)$ in above vertical oscillator is same as when external force of $kZ(\tau)$ is applied to object in horizontal oscillator. [2]

Solution: Fig. 1(a) shows the deflection $X(\tau)$ of object$m$ when the upper end of spring is vertically displaced by displacement $Z(\tau)$. Fig. 1(b) shows the deflection $V(\tau)$ of object$m$ when the upper end of string is not moved vertically $Z(\tau) = 0$.

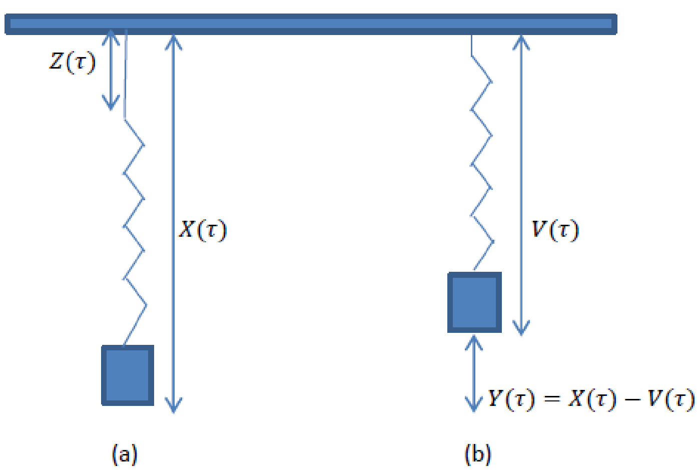

Fig. 1

The differential equations representing the system shown in Fig. 1(a) and Fig. 1(b) are

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$mX''(\tau) = -k[X(\tau) - Z(\tau)] + mg$ (1)

$mV''(\tau) = -kV(\tau) + mg$ (2)

Subtracting (2) from (1)

$$m[X(\tau) - V(\tau)]'' = -k[X(\tau) - V(\tau) - Z(\tau)] \quad (3)$$

As shown in Fig. 1 $Y(\tau) = X(\tau) - V(\tau)$ represent the displacement of object m from the equilibrium position when attached to spring with respect to $Z(\tau) = 0$.

So (3) can be written as $mY''(\tau) = -k[Y(\tau) - Z(\tau)]$. (4)

$$mY''(\tau) = -kY(\tau) + kZ(\tau)$$

Above system can be assumed as forced vibrations in displacement in horizontal oscillator with external force as $kZ(\tau)$.

Illustration3: Considering the oscillator described in Problem1 and 2, the upper end of spring is given initially a vertical displacement $Z(\tau)$ of 1cm downward and kept in the same position for time equal to double time period and comes back to original position. Find $Z(\tau)$. If initial velocity of object is zero, show that it makes two oscillations of 2cm downward and returns to original position and stays thereafter. [2]

Solution: Considering (4) solution of Problem 2

$$mY''(\tau) = -k[Y(\tau) - Z(\tau)] \quad (5)$$

Upper end of string is given displacement in the form

$$Z(\tau) = \begin{cases} 0.01, & 0 \le \tau \le 2T \\ 0, & \tau \ge 2T \end{cases} = 0.25[1 - u(\tau - 2T)] \quad (6)$$

Where T is period of oscillation and $u(\tau - 2T)$ is unit step function.

Given object$m$ initially hangs at rest and no initial displacement given to object m.

$$Y'(0) = Y''(0) = 0 \quad (7)$$

Apply Laplace transform to (5) and using (6), (7)

Let $L_z[Y(\tau)] = \bar{Y}(z)$

$$m[z^2\bar{Y}(z) - zY(0) - Y'(0)] = -k\left[\bar{Y}(z) - 0.01\left(\frac{1-e^{-2Tz}}{z}\right)\right]$$

$$\bar{Y}(z) = \frac{0.01k}{m}\frac{(1-e^{-2Tz})}{z\left(z^2+\frac{k}{m}\right)} \quad (8)$$

Applying Inverse Laplace transform to (8)

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$$Y(\tau) = \frac{0.01k}{m} L_z^{-1}\left[\frac{1}{z\left(z^2+\frac{k}{m}\right)} - \frac{e^{-2Tz}}{z\left(z^2+\frac{k}{m}\right)}\right] \tag{9}$$

$$L_z^{-1}\left[\frac{e^{-2Tz}}{z\left(z^2+\frac{k}{m}\right)}\right] = \frac{m}{k}u(\tau-2T)[1-\cos\omega_0(\tau-2T)] = \frac{m}{k}u(\tau-2T)[1-\cos\omega_0\tau]$$

Using the values of inverse transforms in (9)

$$Y(\tau) = 0.02\sin^2\left(\frac{\pi\tau}{T}\right)[1-u(\tau-2T)]$$

$$Y(\tau) = \begin{cases} 0.02\sin^2\left(\frac{\pi\tau}{T}\right), & 0 \le \tau \le 2T \\ 0, & \tau \ge 2T \end{cases}$$

Hence makes only two oscillations and returns to original position and stays thereafter. Also at $\tau = \frac{T}{2}$, $Y(\tau) = 0.02\ m = 2$ cm

Illustration4: A horizontal oscillator as shown in Fig. 2 is constructed by attaching an object of mass m to coiled spring on one end and to the dashpot on other end. The viscous damping force, with damping coefficient c, is proportional to velocity. The object is given initial velocity v_o and zero initial displacement. Coefficient of damping is given so large that $c^2 > 4km$. Find displacement of object and show that the object keeps moving till the time approaches $\tau = \frac{1}{2a}\log\frac{c+2am}{c-2am}$, where $a = \frac{\sqrt{c^2-4km}}{2m}$ and then it turn and comes back to original position.[2]

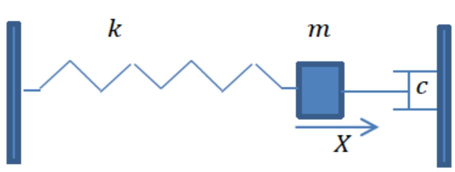

Fig. 2

The differential equation representing above system of motion is

$$mX''(\tau) = -kX(\tau) - cX'(\tau) \tag{10}$$

Applying Laplace transform to (16)

$$m[z^2\bar{X}(z) - zX(0) - X'(0)] = -k\bar{X}(z) - c[z\bar{X}(z) - X(0)] \tag{11}$$

$$\bar{X}(z) = \frac{mv_0}{mz^2+cz+k} = \frac{v_0}{\left(z+\frac{c}{2m}\right)^2 + \frac{4km-c^2}{4m^2}}$$

Given $c^2 > 4km$, above statement can be written as

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$$\bar{X}(z) = \frac{v_0}{\left(z+\frac{c}{2m}\right)^2 - \frac{c^2-4km}{4m^2}}$$

Applying inverse Laplace transform

$$X(\tau) = \frac{v_0}{a}\sinh a\tau\, e^{-\frac{c\tau}{2m}}, \text{ where} a = \frac{\sqrt{c^2-4km}}{2m}$$

(b) $X'(\tau) = \frac{v_0}{a}e^{-\frac{c\tau}{2m}}\left[a\cosh a\tau - \frac{c}{2m}\sinh a\tau\right]$ (12)

Object m will make oscillation in the direction of v_0 and it turns and comes back to original position. The moment when it turns, is velocity will be zero, i.e.$X'(\tau) = 0$

Using (12) $\frac{v_0}{a}e^{-\frac{c\tau}{2m}}\left[a\cosh a\tau - \frac{c}{2m}\sinh a\tau\right] = 0$

$$\coth a\tau = \frac{e^{a\tau}+e^{-a\tau}}{e^{a\tau}-e^{-a\tau}} = \frac{c}{2am}$$

Using Theorem Componendo and dividendo

$$e^{2a\tau} = \frac{c+2am}{c-2am}$$

Hence $\tau = \frac{1}{2a}\log\left(\frac{c+2am}{c-2am}\right)$

Illustration5 : If the horizontal oscillator described in Fig. 2 is given an initial displacement x_0 with zero initial velocity. Find the displacement of object if (a) $c^2 < 4km$ (b)$c^2 = 4km$.[2]

Using the conditions $X'(0) = 0, X(0) = x_0$ in (11)

$$[mz^2+cz+k]\bar{X}(z) = (mz+c)x_0$$

$$\bar{X}(z) = \frac{(mz+c)x_0}{mz^2+cz+k} = \frac{\left(z+\frac{c}{m}\right)x_0}{\left(z+\frac{c}{2m}\right)^2+\frac{4km-c^2}{4m^2}} = \frac{(z+2b)x_0}{(z+b)^2+\left(\frac{k}{m}-b^2\right)} \quad (13)$$

$$X(\tau) = x_0 L_z^{-1}\left[\frac{z+2b}{(z+b)^2+(\omega_0{}^2-b^2)}\right] = x_0 L_z^{-1}\left[\frac{(z+b)+b}{(z+b)^2+(\omega_0{}^2-b^2)}\right]$$

where$2b = c/m$

(a) If $c^2 < 4km$

$$X(\tau) = \frac{x_0 e^{-b\tau}}{\omega_1}[\omega_1\cos\omega_1\tau + b\sin\omega_1\tau] = \frac{x_0 e^{-b\tau}\omega_0}{\omega_1}\cos(\omega_1\tau - \alpha)$$

where$\omega_1 = \sqrt{\frac{k}{m}-b^2} = \sqrt{\omega_0{}^2-b^2}$, $\tan\alpha = \frac{b}{\omega_1}$, $\sin\alpha = \frac{b}{\sqrt{b^2+\omega_1{}^2}} = \frac{b}{\omega_0}$

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(b) If $c^2 = 4km$, then (19) reduces to

$$\bar{X}(z) = \frac{(mz+c)x_0}{mz^2+cz+k} = \frac{\left(z+\frac{c}{m}\right)x_0}{\left(z+\frac{c}{2m}\right)^2} = \frac{(z+2b)x_0}{(z+b)^2} = \frac{(z+b+b)x_0}{(z+b)^2}$$

$$\bar{X}(z) = x_0\left[\frac{1}{z+b} + \frac{b}{(z+b)^2}\right]$$

$$X(\tau) = x_0 e^{-b\tau}(1+b\tau)$$

Flow of Current and Charge in LCR circuit

Illustration6 : Find the flow of current in a LCR circuit when no current flows initially and positive charge q_0 is accumulated on capacitor initially and $> 2\sqrt{\frac{L}{C}}$. Hence show that $I(\tau) \leq 0$and $I(\infty) = 0$. [4]

Solution Let I be the current and Q be charge aggregated at capacitor C . L and R are the inductance and resistance in the electric circuit respectively.

The differential equation representing flow of charge in the circuit is

$$L\frac{dI}{d\tau} + RI + \frac{1}{C}\int_0^\tau I\,d\tau = 0 \tag{14}$$

Given is $Q(0) = q_0 > 0,\ I(0) = 0$

Applying Laplace transform to Eq. (14) and using the above initial conditions

$$L[z\bar{I}(z) - I(0)] + R\bar{I}(z) + \frac{1}{C}\left[\frac{\bar{I}(z)+q_0}{z}\right] = 0$$

Here $\bar{I}(z) = L_z[I(t)]$

$$\left[Lz + R + \frac{1}{zC}\right]\bar{I}(z) = -\frac{q_0}{Cz}$$

$$\bar{I}(z) = -\left[\frac{q_0}{LCz^2+RCz+1}\right] = \frac{-\left(\frac{q_0}{LC}\right)}{\left(z+\frac{R}{2L}\right)^2 - \left(\frac{R^2-\frac{4L}{C}}{4L^2}\right)}$$

If $R > 2\sqrt{\frac{L}{C}}$ and $\alpha = \sqrt{R^2 - \frac{4L}{C}}$

$$I(\tau) = \frac{-2q_0}{\alpha C}e^{-\frac{R\tau}{2L}}\sinh\left(\frac{\alpha\tau}{2L}\right) \tag{15}$$

Since $q_0 > 0$, so $I(\tau) \leq 0$, Eqn. (21) can be written as

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$$I(\tau) = \frac{-q_0}{\alpha C}\left[e^{-(R-\alpha)\frac{\tau}{2L}} - e^{-(R+\alpha)\frac{\tau}{2L}}\right]$$

As $0 < \alpha < R$, so $I(\infty) = 0$

Illustration7 : Electromotive force E_0 is applied to LCR circuit under the conditions that initially no current flows and no charge accumulated on capacitor. Find the flow of current when $\frac{1}{LC} > \left(\frac{R}{2L}\right)^2$. [2]

Solution The differential equation representing flow of charge in the circuit is

$$L\frac{dI}{d\tau} + +QC + RI = E_0$$

$$L\frac{dI}{d\tau} + RI + \frac{1}{C}\int_0^\tau I\, d\tau = E_0$$

Applying Laplace Transform and using the initial conditions $Q(0) = I(0) = 0$

$$L[z\bar{I}(z) - I(0)] + R\bar{I}(z) + \frac{\bar{I}(z)}{zC} = \frac{E_0}{z}$$

$$I(\tau) = \frac{E_0/L}{\left(z + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$I(\tau) = \frac{E_0}{L\omega_1} e^{-b\tau} \sin \omega_1 \tau$$

where$b = \frac{R}{2L}$ and ${\omega_1}^2 = \frac{1}{LC} - b^2 > 0$

Conclusion

The motive of writing this paper is to abridge uses of Laplace Transform in different fields of science and engineering. In this section, some of issues in field of electric circuit and mechanical framework have been settled. The significant favorable position of Laplace Transform is that it makes simple to deal with the arrangement of differential equations and integro-differential equations, which otherwise is very tedious job.

References

[1] B.V. Ramana, Higher Engineering Mathematics, Tata Mc-Graw Hill Publication.

[2] R. V. Churchill, Operational Mathematics, Mc-Graw Hill Publication. [3] Kazem, S., Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform, International Journal of Nonlinear Science, 16 (1): 3-11, (2013).

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[4] Agarana,M. C., & Agboola, O. O., Dynamic Analysis of Damped Driven Pendulum using Laplace Transform Method, International Journal of Mathematics and Computation, Vol. 26 Issue 3, 2015.

[5] Sofi, M. A., & Singh, I., Laplace Transform Method for Solving Electric Circuit Equations, International Journal of Advanced Scientific Research and Management, Vol. 4 Issue 6, June 2019.