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Some New Relations Between Sums of Square Numbers and Triangular Numbers

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Abstract. Let $N_k(n)$ denote the number of ways of expressing $n$as a linear combination of $k$square numbers and $t_k(n)$be the number of ways of writing $n$as a linear combination of $k$triangular numbers, where $n$being a nonnegative integer. Zhi-Hong Sun [*International Journal of Number Theory*, 2018] found several relations between $t$and $N$and posed seven open conjectures. The five conjectures on the ternary case are proved by Kim and Oh [*Journal of Korean Mathematical Society*, 2019] by an elementary method. Xia and Yan [*International Journal of Number Theory,* 2019] proved conjectures 6.1 - 6.6. In this paper, we prove the last conjecture 6.7. We also find some new relations between $N_4(n)$and $t_4(n)$.

Key Words: Sum of squares, sum of triangular numbers, Ramanujan's theta function.

1. Introduction

Let$\mathbb{W}$, $\mathbb{M}$and $\mathbb{Z}$represents the set of whole numbers, positive integers and integers respectively. For $k \in\mathbb{M}$, let $\mathbb{M}^k = \mathbb{M} \times \mathbb{M} \times \cdots \times \mathbb{M}$ (k times)and $\mathbb{Z}^k = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$($k$ times). For $m_i \in\mathbb{M}$and $n \in\mathbb{W}$, define

$$N_k(n) := |\{(y_1, y_2, \cdots, y_k) \in \mathbb{Z}^k | m_1 y_1^2 + m_2 y_2^2 + \cdots m_k y_k^2 = n\}|,$$

$$t_k(n) := \left|\left\{(y_1, y_2, \cdots, y_k) \in \mathbb{Z}^k \middle| m_1 \frac{y_1(y_1+1)}{2} + m_2 \frac{y_2(y_2+1)}{2} + \cdots + m_k \frac{y_k(y_k+1)}{2} = n\right\}\right|,$$

and

$$T_k(n) := \left|\left\{(y_1, y_2, \cdots, y_k) \in \mathbb{M}^k \middle| m_1 \frac{y_1(y_1+1)}{2} + m_2 \frac{y_2(y_2+1)}{2} + \cdots + m_k \frac{y_k(y_k+1)}{2} = n\right\}\right|,$$

where we also set $N_k(0) = T_k(0) = 1$. Since $\frac{y(y+1)}{2} = \frac{(-y)(-y+1)}{2}$, it is clear that

$$t_k(n) = 2^k T_k(n). \tag{1.1}$$

When $m_i = 1$ for all i, then $N_k(n)$ and $T_k(n)$ are usually denoted by $r_k(n)$ and $t_k(n)$, respectively.

It has been found that there exist particular relations between $N_k(n)$and $t_k(n)$with

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$\sum_{i=1}^{m} m_i \geq 9$. Wang and Sun [**5**,**6**] and Sun [**8**] discovered several new relations between $N_4(n)$ and $t_4(n)$.In [**9**], Sun also found several relations between $N_k(n)$ and $t_k(n)$ by using Ramanujan's theta function identities and posed seven open conjectures (Conjecture 6.1 – Conjecture 6.7). In this paper, we prove all of his conjectures by using Ramanujan's theta functions. In the following six theorems, we state conjecture 6.7 of Sun [**9**] and five new results that we prove in this paper [we use the notation $N(a,b,c,d;n)$for $N_4(n)$].

Theorem 1.1.(Conjecture 6.7 in Sun [**9**]) Let $n \in \mathbb{M}$. Then, n is represented by $\frac{z_1(z_1-1)}{2} + \frac{z_2(z_2-1)}{2} + 6\frac{z_3(z_3-1)}{2}$if and only if $n \not\equiv 2.3^{2r-1} - 1(mod\ 3^{2r})$ for $r = 1,2,3$ ...

Theorem 1.2. Let $n\in\mathbb{M}$and $l,k\in\mathbb{N}\cup\{0\}$. Then, we have

$$N(1,1,8l+4,8k+1;8(n+l+k)+7) = t(1,1,8l+4,8k+1;n) \tag{1.2}$$

and

$$N(1,1,8l+4,8k+5;8(n+l+k)+11) = t(1,1,8l+4,8k+5;n). \tag{1.3}$$

Theorem 1.3. Let$n \in \mathbb{M}$. Then,

$$\frac{2}{3}N(1,3,10,14;8n+28) - 2N(1,3,10,14;2n+7) = t(1,3,10,14;n). \tag{1.4}$$

Theorem 1.4. Let $n \in \mathbb{M}$. Then,

$$\frac{2}{3}N(1,2,3,6;8n+12) - 2N(1,2,3,6;2n+3) = t(1,2,3,6;n). \tag{1.5}$$

Theorem 1.5. Let $n \in \mathbb{M}$. Then,

$$\frac{1}{2}N(1,1,4,14;8n+19) = t(1,2,2,14;n).$$

Theorem 1.6. Let $n\in \mathbb{M}$and s,b be any odd positive integer, then

$N(s,s,s,9s;8(n+s)+4) - N(s,s,s,9s;2(n+s)+1) = t(s,s,s,9s;n)$and, for any

$k \in \mathbb{N}$,

$N(b,b,b,9b;4n) = N(b,b,b,9b;4^k n)$.

We organize the remainder of the paper in the following way. Since we use Ramanujan's theta function identities in our proofs, we present basic material on the topic and important lemmas in the next section. In section 3, Theorem 1.1 is proved, whereas the remaining sections are devoted to proving Theorems 1.2–1.6.

2. **Preliminaries**

Ramanujan defined the general theta function$f(a,b)$ by

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$f(a,b) \coloneqq \sum_{k=-\infty}^{\infty} a^{k(k+1)/2}\, b^{k(k-1)/2}$, $|ab| < 1$.

From here onwards and in this complete paper, we will use the notation $g(a,b)$ in place of $f(a,b)$.

Three special cases arise from the definition of $g(a,b)$, given as follows:

$$\varphi(q) \coloneqq g(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{2.1}$$

$$\psi(q) \coloneqq g(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.2}$$

and

$$g(-q) \coloneqq g(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = (q;q)_\infty, \tag{2.3}$$

where the above product representations arises from the Jacobi's TPI taken from [**1**, p. 35, Entry 19].

$$g(r,s) = (-r;rs)_\infty(-s;rs)_\infty(rs;rs)_\infty, \tag{2.4}$$

where the following standard notation is used.

$$(a;q)_\infty = \prod_{j=0}^{\infty}\left(1 - aq^j\right).$$

Also set

$$\chi(q) \coloneqq (-q;q^2)_\infty. \tag{2.5}$$

From the definitions of theta function$\varphi(q)$ and theta function$\psi(q)$, it is clear that

$$\sum_{n=0}^{\infty} N(a_1,a_2,\dots,a_k;n)\, q^n = \varphi(q^{a_1})\varphi(q^{a_2}) \dots \dots \varphi(q^{a_k}) \tag{2.6}$$

and

$$\sum_{n=0}^{\infty} T(a_1,a_2,\dots,a_k;n)\, q^n = \varphi(q^{a_1})\varphi(q^{a_2}) \dots \dots \varphi(q^{a_k}). \tag{2.7}$$

Some preliminary identities of$g(a,b)$are stated in the lemma below.

Lemma 2.1.(Berndt [**1**, pp. 45–46, Entries 29 and 30]) We have

$$g(e,f) + g(-e,-f) = 2g(e^3 f, ef^3), \tag{2.8}$$

$$g(e,f) - g(-e,-f) = 2eg\left(\frac{f}{e}, e^5 f^3\right), \tag{2.9}$$

$$g(e,f)g(r,s) + g(-e,-f)g(-r,-s) = 2g(er,fs)g(es,fr), \tag{2.10}$$

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$$g(e,f)g(r,s)-g(-e,-f)g(-r,-s)=2eg(\tfrac{f}{r},er^2s)g(\tfrac{f}{s},ers^2), \qquad (2.11)$$

Setting $r=s=e=f=q$ in (2.8) – (2.11) and using (Berndt [**1**, p. 40, Entry 25]) implies the following 2-dissections of $\varphi(q)$ and $\varphi^2(q)$.

Lemma 2.2. We have

$\varphi(q)=\varphi(q^4)+2q\psi(q^8)$, (2.12) $\varphi^2(q)=\varphi^2(q^2)+4q\psi^2(q^4)$.
(2.13)

In the next lemma, we recall 2- and 3-dissections of $\psi(q)$ and $\phi(q)$ from Berndt's book [**1**, p. 49, Corollary].

Lemma 2.3. We have

$\varphi(q)=\varphi(q^9)+2qf(q^3,q^{15})$, (2.14) $\psi(q)=f(q^6,q^{10})+qf(q^2,q^{14})$, (2.15)

$$\psi(q)=f(q^3,q^6)+q\psi(q^9). \qquad (2.16)$$

Some 2-dissections of products of theta functions are recorded in the following lemma.

Lemma 2.4. The following identities hold

$$\varphi(q)\varphi(q^3)=\varphi(q^4)\varphi(q^{12})+2q\psi(q^2)\psi(q^6)+4q^4\psi(q^8)\psi(q^{24}), \qquad (2.17)$$
$$\psi(q)\psi(q^3)=\psi(q^4)\varphi(q^6)+q\varphi(q^2)\psi(q^{12}), \qquad (2.18)$$
$$\psi(q^3)\psi(q^5)=\psi(q^8)\varphi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\varphi(q^4)\psi(q^{120}), \quad (2.19)$$
$$\psi(q)\psi(q^7)=\psi(q^8)\varphi(q^{28})+q\psi(q^2)\psi(q^{14})+q^6\varphi(q^4)\psi(q^{56}), \quad (2.20)$$
$$\psi(q)\psi(q^{15})=\psi(q^6)\psi(q^{10})+q\varphi(q^{20})\psi(q^{24})+q^3\varphi(q^{12})\psi(q^{40}). \qquad (2.21)$$

Proof. Identity (2.17) follows by setting (μ,ν) = (2,1) in [**1**, p. 68, eq. (36.2)], and then employing (**2.12**) and (**2.13**). Identities (2.18), (2.20) and (2.19) follow from (36.8) of [**1**, p. 69] by setting (μ,ν) = (2,1), (μ,ν) = (4,3) and (μ,ν) = (4,1), respectively.
To prove, identity (2.21), we first note, by (2.12), that

$$\varphi(q^3)\varphi(q^5)-\varphi(-q^3)\varphi(-q^5)=4q^3(\varphi(q^{20})\psi(q^{24})+q^2\varphi(q^{12})\psi(q^{40})).$$

But, by [**?**, p. 377, Entry 9(iii)], we also have

$$\varphi(q^3)\varphi(q^5)-\varphi(-q^3)\varphi(-q^5)=2q^2(\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})).$$

From the last two equations,

$$\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})=2q(\varphi(q^{20})\psi(q^{24})+q^2\varphi(q^{12})\psi(q^{40})).$$

On the other hand, from [**1**, p. 377, Entry 9(iv)], we recall that

$$\psi(q)\psi(q^{15})+\psi(-q)\psi(-q^{15})=2\psi(q^6)\psi(q^{10}).$$

Adding the previous two identities, we readily arrive at (2.21).

3. **Proof of Theorems 1.1**

One way of the theorem has already been proved by Sun [**9**], namely, if$n=\frac{z_1(z_1-1)}{2}+\frac{z_2(z_2-1)}{2}+6\frac{z_3(z_3-1)}{2}$, then$n\not\equiv 2.3^{2r-1}-1(mod\ 3^{2r})forr=1,2,3,....$
Therefore, we need to show only the other way, that is, if $n\equiv 2.3^{2r-1}-1(mod\ 3^{2r})$, then $T(1,1,6;n)=0$. This is equivalent to showing that if$n=3^{2r}k+2.3^{2r-1}-1$, for some integer $k$and $r=1,2,3,\ldots..$, then

$$T(1,1,6;n)=0. \tag{3.1}$$

Note that, when $r=1$, then$n=9k+5=3(3k+1)+2$.
From (2.7), we have

$$\sum_{n=0}^{\infty}T(1,1,6;n)q^n=\psi^2(q)\psi(q^6).$$

With the use of (2.16) and extraction of terms involving q^{3n+2} from both sides, we find from the above equation that

$$\sum_{n=0}^{\infty}T(1,1,6;3n+2)q^n=\psi(q^2)\psi^2(q^3). \tag{3.2}$$

Again, employing (2.16) in the above equation, and comparing the coefficients of q^{3n+1} from both sides, we find

$$T(1,1,6;3(3n+1)+2)=0. \tag{3.3}$$

which is the case$r=1$ of (3.1).
Again, using (2.16) in (3.2), and comparison of the terms involving$q^{3n+2}$from both sides gives

$$\sum_{n=0}^{\infty}T(1,1,6;3(3n+2)+2)q^n=\psi^2(q)\psi(q^6)=\sum_{n=0}^{\infty}T(1,1,6;n)q^n,$$

from which we readily arrive at

$$T(1,1,6;3^2n+2.3+2)=T(1,1,6;n),$$

that is,

$$T(1,1,6;3^2n+3.3-1)=T(1,1,6;n).$$

Replacing $n$by$3(3n+1)+2$ in the above and then using (3.3), we have

$$T(1,1,6;3^2(3(3n+1)+2)+3.2-1)=T(1,1,6;3(3n+1)+2)=0$$

that is,

$$T(1,1,6;3^4n+2.3^3-1)=0,$$

which is the case$r=2$ of (3.1).
Now, (3.1) can be easily proved for$r=1,2,3,\ldots\ldots$, by mathematical induction.
From Baruah, Cooper and Hirschhorn [**7**, Theorem 1.4], we note that

$$N(1,1,6;8n+8)-N(1,1,6;2n+2)=8T(1,1,6,n).$$

From Theorem 1.1 and the above we have the following interesting result.

Corollary 3.1. If $n \equiv 2.3^{2r-1} - 1 (mod\ 3^{2r})$ for $r = 1,2,3, \dots \dots$, then

$N(1,1,6; 8n + 8) = N(1,1,6; 2n + 2)$.

Remark 3.2. *The following general result can be verified by proceeding in a similar way as in the above proof of Theorem 1.1. If* $n \equiv 2.3^{2r-1} - 1 (mod\ 3^{2r})$ *and* k *is a non-negative integer, then* $T(1,1,9k + 6; n) = 0$.

4. Proof of Theorem 1.2

We note from (2.6) that

$$\sum_{n=0}^{\infty} N(1,1,8l + 4,4s + 1; n)q^n = \varphi^2(q)\varphi(q^{8l+4})\varphi(q^{4s+1})$$

Using (2.12) and (2.13) in the last equation, extracting the odd powers of q, we obtain

$$\sum_{n=0}^{\infty} N(1,1,8l + 4,4s + 1; 2n + 1)q^n$$
$$= 4\psi^2(q^2)\varphi(q^{4l+2})\varphi(q^{8s+2}) + 2q^{2s}\psi(q^{16s+4})\varphi^2(q)\varphi(q^{4l+2}).$$

Again, application of (2.13) and extraction of the odd powers of q implies from above that

$$\sum_{n=0}^{\infty} N(1,1,8l + 4,4s + 1; 4n + 3)q^n = 8q^s\psi(q^{8s+2})\psi^2(q^2)\varphi(q^{2l+1}) = 8q^s\psi(q^{8s+2})\psi^2(q^2)(\varphi(q^{8l+4}) + 2q^{2l+1}\psi(q^{16l+8})), \tag{4.1}$$

where the last equality is due to (2.12).

Now if s is even, say, $s = 2k$, then extracting the odd powers of q from both sides of (4.1), we arrive that

$$\sum_{n=0}^{\infty} N(1,1,8l + 4,8k + 1; 8n + 7)q^n = 16q^{l+k}\psi^2(q)\psi(q^{8l+4})\psi(q^{8k+1}). \tag{4.2}$$

On the other hand, if s is odd, say, $s = 2k + 1$, then extracting the even powers of q from both sides of (4.1), we have

$$\sum_{n=0}^{\infty} N(1,1,8l + 4,8k + 5; 8n + 3)q^n = 16q^{l+k+1}\psi^2(q)\psi(q^{8l+4})\psi(q^{8k+5}). \tag{4.3}$$

Comparing the terms involving q^{n+l+k} and $q^{n+l+k+1}$, from both sides of (4.2) and (4.3), respectively, we arrive at

$$N(1,1,8l + 4,8k + 1; 8(n + l + k) + 7) = 16T(1,1,8l + 4,8k + 1; n) \text{ and}$$

$$N(1,1,8l + 4,8k + 5; 8(n + l + k) + 11) = 16T(1,1,8l + 4,8k + 5; n),$$

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which, by (1.1), are equivalent to (1.2) and (1.3), respectively. This proves Theorem 1.2.

5. Proof of Theorem 1.3

From (2.6) and (2.17), we have

$$\sum_{n=0}^{\infty} N(1,3,10,14;n)q^n = \varphi(q)\varphi(q^3)\varphi(q^{10})\varphi(q^{14}) = \varphi(q^{10})\varphi(q^{14})(\varphi(q^4)\varphi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24})).$$

Extracting the even and odd powers of q separately, we find that

$$\sum_{n=0}^{\infty} N(1,3,10,14;2n)q^n = \varphi(q^5)\varphi(q^7)\{\varphi(q^2)\varphi(q^6) + 4q^2\psi(q^4)\psi(q^{12})\} \quad (5.1)$$

and

$$\sum_{n=0}^{\infty} N(1,3,10,14;2n+1)q^n = 2\psi(q)\psi(q^3)\varphi(q^3)\varphi(q^5)\varphi(q^7). \quad (5.2)$$

Use of (2.12) in (5.1), and extraction of the even powers of q from both sides gives

$$\sum_{n=0}^{\infty} N(1,3,10,14;4n)q^n = \big(\varphi(q^{10})\varphi(q^{14}) + 4q^6\psi(q^{20})\psi(q^{28})\big)\big(\varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6)\big),$$

which, by (2.17), implies

$$\sum_{n=0}^{\infty} N(1,3,10,14;4n)q^n = \big(\varphi(q^{10})\varphi(q^{14}) + 4q^6\psi(q^{20})\psi(q^{28})\big)(\varphi(q^4)\varphi(q^{12}) + 6q\psi(q^2)\psi(q^6) + 4q^4\psi(q^2)\psi(q^6)),$$

Extraction of the coefficients ofq^{2n+1}implies from above that

$$\sum_{n=0}^{\infty} N(1,3,10,14;8n+4)q^n = 6\psi(q)\psi(q^3)\varphi(q^5)\varphi(q^7) + 24q^3\psi(q)\psi(q^3)\psi(q^{10})\psi(q^{14}),$$

which can be recast with the aid of (5.2) and (2.7) as

$$\sum_{n=0}^{\infty} \big(N(1,3,10,14;8n+4)q^n - 3N(1,3,10,14;2n+1)\big)q^n = 24\sum_{n=0}^{\infty} T(1,3,10,14;n)q^{n+3}.$$

Comparing the coefficients ofq^{n+3}on both sides of the above and then using (1.1), we prove (1.4).

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6. Proofs of Theorems 1.4–1.6

The proof is similar to that of Theorem 1.3 given in the previous section. We apply (2.17) to the generating function

$$\sum_{n=0}^{\infty} N(1,2,3,6;n)q^n = \varphi(q)\varphi(q^2)\varphi(q^3)\varphi(q^6),$$

and then extract the even and/or odd powers of q, to find that

$$\sum_{n=0}^{\infty} N(1,2,3,6;2n)q^n = \varphi(q)\varphi(q^3)\big(\varphi(q^2)\varphi(q^6) + 4q^2\psi(q^4)\psi(q^{12})\big),$$

$$\sum_{n=0}^{\infty} N(1,3,10,14;2n+1)q^n = 2\psi(q)\psi(q^3)\varphi(q)\varphi(q^3),$$

$$\sum_{n=0}^{\infty} N(1,2,3,6;4n)q^n$$
$$= (\varphi(q^2)\varphi(q^6) + 4q^2\psi(q^4)\psi(q^{12}))\big(\varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6)\big),$$

$$\sum_{n=0}^{\infty} N(1,2,3,6;8n+4)q^n = 6\psi(q)\psi(q^3)\big(\varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6)\big).$$

Thus,

$$\sum_{n=0}^{\infty} (N(1,2,3,6;8n+4) - 3N(1,2,3,6;2n+1))q^n = 24q\psi(q)\psi(q^2)\psi(q^3)\psi(q^6)$$

$$= 24\sum_{n=0}^{\infty} T(1,2,3,6:n)q^{n+1}.$$

Equating the coefficients of$q^{n+1}$and use of (1.1) implies (1.5) to complete the proof.

The proof of Theorem 1.5 and the proof of Theorem 1.6 are almost same, so we omit them.

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