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C_2-Sequence Spaces In 2-Normed Spaces and Ideal Convergence

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Abstract

In this paper, some C_2-sequence spaces are defined using an Orlicz function and ideal convergence in 2-mnormed spaces and examined their topological and algebraic properties.

1. Introduction

There are two numbers systems viz. Quaternions and bicomplex numbers as the generalization of complex numbers but their algebraic structures are entirely different. The Quaternions forms a non-commutative division algebra but the set of bicomplex numbers forms a commutative ring with zero divisors. The concept of bicomplex numbers was introduced by Price [6]. The C_2, C_0 and C_1 denote the set of bicomplex numbers [6], complex numbers and real numbers, respectively.

Definition 1.1. Bicomplex Space: The set of bicomplex number is defined as (cf. [6-8])

$$C_2 := \{ d_1 + j_1 d_2 + j_2 d_3 + j_1 j_2 d_4 : d_k \in C_0, 1 \leq k \leq 4 \} = \{w_1 + j_2 w_2 : w_1, w_2 \in C_1\},$$

where $j_1^2 = j_2^2 = -1,\ j_1 j_2 = j_2 j_1$.

The set C_2 has only two non-trivial idempotent elements given by e_1 and e_2 having values $e_1 = \frac{1 + j_1 j_2}{2}$ and $e_2 = \frac{(1 - j_1 j_2)}{2}$.

Note that $e_1 + e_2 = 1$ and $e_1 . e_2 = 0$.

The number $\xi = w_1 + j_2 w_2$ can also be written as [7]**:**

$$\xi = w_1 + j_2 w_2 = {}^1\xi\, e\_1 + {}^1\xi\, e_2,$$

where ${}^1\xi = w_1 - j_1 w_2$ and ${}^2\xi = w_1 + j_1 w_2$. This representation is a unique for each bicomplex number with respect to linear combination of e_1 and e_2.

The terms ${}^1\xi$ and ${}^2\xi$ are known as idempotent components of ξ, and ${}^1\xi\, e_1 + {}^1\xi\, e_2$ is called as idempotent representation of bicomplex number ξ. The auxiliary spaces B_1 and B_2 are defined as $B_1 = \{{}^1\xi : \xi \in C_2\}$ and $B_2 = \{ {}^2\xi : \xi \in C_2\}$.

The norm in C_2 is defined as follows:

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$$\|\xi\| = \sqrt{d_1^2 + d_2^2 + d_3^2 + d_4^2} = \sqrt{|w_1|^2 + |w_2|^2} = \sqrt{\frac{\left|{}^1\xi\right|^2 + \left|{}^2\xi\right|^2}{2}}$$

The following inequality is the best possible relation for the norm on C_2:

$$\|\xi \cdot \eta\| \leq \sqrt{2}\, \|\xi\|\, \|\eta\|$$

For this reason, we call C_2 as modified complex Banach algebra [6].

Throughout the paper, the ω_4, c, c_0 and $\ell_{C_2}^{\infty}$ denote the space of all C_2-sequences, convergent sequences, null sequences and all bounded sequences. We denote the zero sequence $(0, 0, 0, \ldots, 0, \ldots)$ as π and $s = \{s_k\}$ is a sequence of positive real numbers and $\{ s_k^{-1}\} = \{q_k\}$.

The concept of ideal convergence was introduced by the Kostyrko et al. [1]. An ideal on N is a collection of subsets of N is closed under the finite unions and subsets of its elements. It is considered as the generalization of statistical [2].

Definition 1.2. Orlicz Function: An Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is defined as:

- **(i)** It is continuous on $[0, \infty)$
- **(ii)** It is a convex function
- **(iii)** It is a non-decreasing function on $[0, \infty)$
- **(iv)** $M(0) = 0$,
- **(v)** $M(a) > 0$ for $a > 0$,
- **(vi)** $M(a) \rightarrow \infty$ as $a \rightarrow \infty$.

The Orlicz function, M is called *modulus function* if it satisfies the property:

$$M(a + b) \leq M(a) + M(b),$$

Note that if M is an Orlicz function , then

$$M(\lambda a) \leq \lambda M(a), \quad \forall\, \lambda \text{ with } 0 < \lambda < 1.$$

Let $(X, \|.\|)$ be a normed space. A sequence $\{a_n\}$ in X is called as statistically convergent to $a \in X$, if the set $K(\epsilon) = \{n \in N : \|a_n - a\| \geq \epsilon\}$ has natural density equal to zero for every $\epsilon > 0$.

Definition 1.3. Ideal: A family $B \subset 2^Y$ is called an idea if

- **(i)** $\phi \notin B$
- **(ii)** $R, S \in B \;\Rightarrow\; R \cup S \in B$
- **(iii)** $R \in B, S \subset R \;\Rightarrow\; S \in B$.

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An ideal B is said to be *admissible ideal* of Y, if $\{x\} \in B$ for each $x \in Y$, [9, 10]. Given $B \subset 2^Y$ is a nontrivial ideal in N. The sequence $\{a_n\} \in X$ is said to be B-convergent to $x \in X$ if for each $\epsilon > 0$, the set $K(\epsilon) = \{n \in N : \|a_n - a\| \geq \epsilon\}$ belong to B (cf. [1, 3]).

Recently, a lot of researchers have started studying the concepts of summability in these types of nonlinear spaces (cf. [8-10]). Let X be a real finite dimensional vector space. The 2-norm on X is a function $\|.,.\| : X \times X \to R$, having the following properties:

- **(i)** $\|a, b\| = 0$ if and only if a and b are linearly dependent,
- **(ii)** $\|a, b\| = \|b, a\|$,
- **(iii)** $\|\gamma a, b\| = |\gamma| \|a, b\|, \gamma \in R$,
- **(iv)** $\|a, b + c\| \leq \|a, b\| + \|a, c\|$.

The pair $(X, \|.,.\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent X.

In this paper, we discuss some C_2-sequence spaces by using ideal convergence and Orlicz function in 2-normed spaces.

2. C_2-Sequence Spaces

Let $\Delta = \{\delta_n\}$ be a sequence of positive numbers such that it is non-decreasing and unbounded with the condition $\delta_1 = 0$. Let $(X, \|.\|)$ be a 2-normed space, J be an ideal of N, and M be an Orlicz function. Also assume that $q = \{q_k\}$ be a bounded real sequence with $q_k > 0$. Throughout the paper, ω_4 denote the linear space of all C_2-sequences defined over $(C_2, \|.,.\|)$. For some $K > 0, L \in C_2$ and each $z \in C_2$, we defined some C_2-sequence spaces:

(i) $A^J(C_2, \|.,.\|, M, \delta, q) = \left\{x \in \omega_4 : \forall \epsilon > 0 \left\{n \in N: \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k - L}{K}, z\right\|\right)\right]^{q_k}\right\} \geq \epsilon\right\} \in I$,

(ii) $A_0^J(C_2, \|.,.\|, M, \delta, q) = \left\{x \in \omega_4 : \forall \epsilon > 0 \left\{n \in N: \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k}\right\} \geq \epsilon\right\} \in I$,

(iii) $A_\infty(C_2, \|.,.\|, M, \delta, q) = \left\{x \in \omega_4 : \exists T > 0 \; s.t. \; \sup_{n \in N} \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} < T\right\}$,

(iv) $A_\infty^J(C_2, \|.,.\|, M, \delta, q) = \left\{x \in \omega_4 : \exists T > 0 \, s.t. \left\{n \in N: \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k}\right\} \geq T\right\} \in I$,

where $J_n = [n - \delta_n + 1, n]$. we shall use the following inequality in proving the results [11]:

$$0 \leq q_k \leq \sup q_k = H, D = \max\{1, 2^{H-1}\},$$

Then,

$$|z_k + w_k|^{q_k} \leq D\left(|z_k|^{q_k} + |w_k|^{q_k}\right),$$

$\forall k, z_k, w_k \in C_1$. Further, $|z_k|^{q_k} \leq \max\{1, |z_0|^H\}$, for some $z_0 \in C_1$.

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Theorem 2.1: The set $A_0^J(C_2, \|.,.\|, M, \delta, q)$ is a linear space over C_1.

Proof: Let $\xi, \eta \in A_0^J(C_2, \|.,.\|, M, \delta, q)$ and $\alpha, \beta \in C_1$, so that

$$\left\{ n \in N : \frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\xi_k}{K_1}, z \right\| \right) \right]^{q_k} \geq \epsilon \right\} \in J$$

for some $K_1 > 0$,

$$\left\{ n \in N : \frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\eta_k}{K_2}, z \right\| \right) \right]^{q_k} \geq \epsilon \right\} \in J$$

for some $K_2 > 0$, Then,

$$\frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\alpha \xi_k + \beta \eta_k}{|\alpha| K_1 + |\beta| K_2}, z \right\| \right) \right]^{q_k}$$

$$\leq D.\frac{1}{\delta_n} \sum_{k \in J_n} \left[\frac{|\alpha|}{|\alpha| K_1 + |\beta| K_2} M \left(\left\| \frac{\xi_k}{K_1}, z \right\| \right) \right]^{q_k} + D.\frac{1}{\delta_n} \sum_{k \in J_n} \left[\frac{|\beta|}{|\alpha| K_1 + |\beta| K_2} M \left(\left\| \frac{\eta_k}{K_2}, z \right\| \right) \right]^{q_k}$$

$$\leq DF.\frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\xi_k}{K_1}, z \right\| \right) \right]^{q_k} + DF.\frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\eta_k}{K_2}, z \right\| \right) \right]^{q_k},$$

where

$$F = \max \left[1, \left(\frac{|\alpha|}{|\alpha| K_1 + |\beta| K_2} \right)^H, \left(\frac{|\beta|}{|\alpha| K_1 + |\beta| K_2} \right)^H \right].$$

Hence,

$$\left\{ n \in N : \frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\alpha \xi_k + \beta \eta_k}{|\alpha| K_1 + \beta K_2}, z \right\| \right) \right]^{q_k} \geq \epsilon \right\}$$

$$\subseteq \left\{ n \in N : DF \frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\xi_k}{K_1}, z \right\| \right) \right]^{q_k} \geq \epsilon \right\} \cup \left\{ n \in N : DF \frac{1}{\delta_n} \sum_{k \in J_n} \left[M \left(\left\| \frac{\eta_k}{K_2}, z \right\| \right) \right]^{q_k} \geq \epsilon \right\}.$$

Corollary 2.1: $A^J(C_2, \|.,.\|, M, \delta, q), A_\infty(C_2, \|.,.\|, M, \delta, q)$ and $A_\infty^J(C_2, \|.,.\|, M, \delta, q)$ are linear spaces.

Theorem 2.2: The space $A_\infty(C_2, \|.,.\|, M, \delta, q)$ is a paranormed space with respect to paranorm function defined as

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$$g_n(x) = \inf\left\{K^{\frac{q_n}{H}} : K > 0 \; s.t. \; \left(\sup \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k}\right)^{1/H} \leq 1, \forall\, z \in X\right\}.$$

Proof: Clearly, $g_n(\theta) = 0$ and $g_n(-x) = g(x)$.

Now, let $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ in $A_\infty(C_2, \|.,.\|, M, \delta, q)$. Define

$$E(\xi) = \left\{K > 0 : \sup \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} \leq 1, \forall\, z \in X\right\},$$

$$E(\eta) = \left\{K > 0 : \sup \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\eta_k}{K}, z\right\|\right)\right]^{q_k} \leq 1, \forall\, z \in X\right\}.$$

Let $K_1 \in E(\xi)$ and $K_2 \in E(\eta)$, then if $K = K_1 + K_2$, we have

$$\sup \frac{1}{\delta_n} \sum_{k \in J_n} M\left(\left\|\frac{\xi_k + \eta_k}{K}, z\right\|\right)$$
$$\leq \frac{K_1}{K_1 + K_2} \sup \frac{1}{\delta_n} \sum_{k \in J_n} M\left(\left\|\frac{\xi_k}{K_1}, z\right\|\right) + \frac{K_1}{K_1 + K_2} \sup \frac{1}{\delta_n} \sum_{k \in J_n} M\left(\left\|\frac{\eta_k}{K_1}, z\right\|\right).$$

Thus,

$$\sup_n \frac{1}{\delta_n} \sum_{k \in J_n} M\left(\left\|\frac{\xi_k + \eta_k}{K_1 + K_2}, z\right\|\right)^{q_k} \leq 1$$

and

$$\begin{aligned} g_n(\xi_k + \eta_k) &\leq \inf\left\{(K_1 + K_2)^{\frac{q_n}{H}} : K_1 \in E(\xi), K_2 \in E(\eta)\right\} \\ &\leq \inf\left\{K_1^{\frac{q_n}{H}} : K_1 \in E(\xi)\right\} + \inf\left\{K_2^{\frac{q_n}{H}} : K_2 \in E(\eta)\right\} \\ &= g_n(\xi) + g_n(\eta). \end{aligned}$$

The continuity of scalar multiplication is straightforward. Hence proved.

Corollary 2.2: For some $F \in J$, we have

$$A_\infty(C_2, \|.,.\|, M, \delta, p) = \left\{\xi = \{\xi_k\} \in \omega_4 : \exists K > 0;\ \sup_{n \in G} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k}{E}, z\right\|\right)\right]^{q_k} \leq K, E > 0\right\},$$

where $G = N/F$, for each $z \in C_2$. It is a subspace of $A_\infty^J(C_2, \|.,.\|, M, \delta, q)$.

Definition 2.1: A sequence space is known as *solid space* if $\{\alpha_k \xi_k\} \in s$, whenever $\{\xi_k\} \in s$ for any sequence $\{\alpha_k\}$ of scalars with $|\alpha_k| \leq 1$, for all $k \in N$.

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Theorem 2.3: The spaces $A_0^J(C_2, \|.,.\|, M, \delta, q)$ and $A_\infty^J(C_2, \|.,.\|, M, \delta, q)$ are solid spaces.

Proof: We shall prove the result for $A_0^J(C_2, \|.,.\|, M, \delta, q)$ only.

Suppose that $\{\xi_k\} \in A_0^J(C_2, \|.,.\|, M, \delta, q)$ and (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$, $\forall\, k \in N$. Then we have

$$\left\{n \in N : \frac{1}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\alpha_k \xi_k}{K}, z\right\|\right)\right]^{q_k} \geq \epsilon\right\} \subseteq \left\{n \in N : \frac{D}{\delta_n} \sum_{k \in J_n} \left[M\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} \geq \epsilon\right\} \in J,$$

where $D = \max\{1, |\alpha_k|^H\}$.

Hence, $\{\alpha_k \xi_k\} \in A_0^J(C_2, \|.,.\|, M, \delta, q)$.

Theorem 2.4: Assume M_1, M_2 be two Orlicz functions. Then, we have

(i) $A_0^J(C_2, \|.,.\|, M_1, \delta, q) \subseteq A_0^J(C_2, \|.,.\|, M_1 o M_2, \delta, q)$ provided (q_k) is such $H_0 = \inf q_k > 0$.

(ii) $A_0^J(C_2, \|.,.\|, M, \delta, q) \cap A_0^J(C_2, \|.,.\|, M_2, \delta, q) \subseteq A_0^J(C_2, \|.,.\|, M_1 + M_2, \delta, q)$.

Proof: (i) For given $\epsilon > 0$, choose $\epsilon_1 > 0$ such that $\max\{\epsilon_1^H, \epsilon_1^{H_0}\} < \epsilon$. By using the continuity of M_1 let $0 < \lambda < 1$ with $0 < t < \lambda$ such that $M(t) < \epsilon_0$.

Let $(\xi_k) \in A_0^J(C_2, \|.,.\|, M_1, \delta, q)$.

Then,

$$E(\lambda) = \left\{n \in N : \frac{1}{\delta_n} \sum_{k \in J_n} \left[M_1\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} \geq \lambda^H\right\} \in J.$$

Thus, if $n \notin E(\lambda)$, then

$$\frac{1}{\delta_n} \sum_{k \in J_n} \left[M_2\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} < \lambda^H$$

i.e.,

$$\left[M_2\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} < \lambda^H, \forall\, k \in J_n$$

Thus,

$$M_2\left(\left\|\frac{\xi_k}{K}, z\right\|\right) < \lambda$$

Hence, by the continuity of Orlicz function M, we have

$$M_1\left[M_2\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right] < \epsilon_0, \quad \forall\, k \in J_n$$

which implies that

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$$\sum_{k \in J_n} \left[M_1 o M_2 \left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} < \epsilon\, \delta_n$$

$$\Rightarrow \frac{1}{\delta_n} \sum_{k \in J_n} \left[M_1 \left(M_2 \left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right)\right]^{q_k} < \delta_n.$$

Therefore,

$$\left\{n \in N : \frac{1}{\delta_n} \sum_{k \in J_n} \left[M_1 \left(M_2 \left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right)\right]^{q_k} \geq \epsilon\right\} \subset E(\lambda)$$

Hence,

$$\left\{n \in N : \frac{1}{\delta_n} \sum_{k \in J_n} \left[M_1 \left(M_2 \left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right)\right]^{q_k} \geq \epsilon\right\} \in J.$$

(iii) Suppose that $(\xi_k) \in A_0^J(C_2, \|.,.\|, M_1, \delta, q) \cap A_0^J(C_2, \|.,.\|, M_2, \delta, q)$, then,

$$\frac{1}{\delta_n}\left[(M_1 + M_2)\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} \leq D\, \frac{1}{\delta_n}\left[M_1\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k} + D\frac{1}{\delta_n}\left[M_2\left(\left\|\frac{\xi_k}{K}, z\right\|\right)\right]^{q_k}.$$

Clearly,

$$(\xi_k) \in A_0^J(C_2, \|.,.\|, M_1 + M_2, \delta, p).$$

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