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Some New Identities Involving Relation Between Chebyshev Polynomials and Lucas Number

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Abstract: In this paper we have proven some new identities involving Chebyshev polynomials and Lucas numbers.

1 INTRODUCTION

Chebyshev polynomials are applied in several parts of numerical analysis and more precisely, we can say that, in the applications of mathematics. It's very difficult to ignore Chebyshev polynomials as they are seen almost in every branch of mathematics.We know very less about the relation between the Chebyshev polynomials and some well-known sequences. In this paper we have discussed some of them.

1.1 Chebyshev polynomials

The Chebyshev polynomial of the first kind is defined bythe recurrence relation

$$T_{n+2}(z) = 2zT_{n+1}(z) - T_n(z) \tag{1.1}$$

For integer $n \geq 0$, with initial condition

$T_0(z) = 1, T_1(z) = z.$

$T_n(z)$ can be explicitly define as

$$T_n(z) = \frac{1}{2}\left(\left(z - \sqrt{z^2-1}\right)^n + \left(z + \sqrt{z^2-1}\right)^n\right) \tag{1.2}$$

and

The Chebyshev polynomial of thesecond kind is defined by the recurrence relation

$$U_{n+2}(z) = 2zU_{n+1}(z) - U_n(z) \tag{1.3}$$

For integer $n \geq 0$

with initial condition

$U_0(z) = 1 , U_1(z) = 2z,$

$U_n(z)$ can be explicitly define as

$$U_n(z) = \frac{\left(z + \sqrt{z^2-1}\right)^{n+1} - \left(z - \sqrt{z^2-1}\right)^{n+1}}{2\sqrt{z^2-1}} \tag{1.4}$$

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1.2 Lucas Sequence

Lucas sequence $\{L_n\}_{n\in N}$ is defined by

$$L_{n+1} = L_n + L_{n-1} \tag{1.5}$$

For integer $n \geq 1$ with initial condition

$$L_0 = 2\,, L_1 = 1$$

Binet's formula in Lucas number theory allows us to express theLucas number in function of roots

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

of the characteristic equation

$$r^2 = r + 1$$

associated to recurrence relation (1.5) is

$$L_n = r_1^n + r_2^n \tag{1.6}$$

Theorem 1: let k be any positive integer,n be non-negative even integer,L_n is the n^{th} Lucas numberand $U_n(z)$is define by (1.3), $U_n^k(z)$denote k^{th} derivatives of $U_n(z)$ with respect to z. then

$$\sum_{b_1+b_2+\cdots+b_{k+1}=n+k+1} L_{18(b_1)}L_{18(b_2)}\dots L_{18(b_{k+1})} = \frac{2}{k!}\sum_{h=0}^{k+1}(-2889)^h\binom{k+1}{h}U^{(k)}_{n+2k+1-h}(2889) \tag{1.7}$$

Proof: If$T_n(z)$ isdefined by (1.1) then by $[2]$

$$\sum_{b_1+b_2+\cdots+b_{k+1}=n+k+1} T_{(b_1)}(z)T_{(b_2)}(z)\dots T_{(b_{k+1})}(z) = \frac{1}{2^k k!}\sum_{h=0}^{k+1}(-z)^h\binom{k+1}{h}U^{(k)}_{n+2k+1-h}(z) \tag{1.8}$$

Taking $z = 2889$,using equations(1.2), (1.5) and (1.6) we obtain

$$T_n(2889) = \frac{1}{2}L_{18n} \tag{1.9}$$

By using equations (1.8) , (1.9) and solving we obtain

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$$\sum_{b_1+b_2+\cdots+a_{k+1}=n+k+1} L_{18(b_1)}L_{18(b_2)}\ldots L_{18(b_{k+1})}$$
$$= \frac{2}{k!}\sum_{h=0}^{k+1}(-2889)^h\binom{k+1}{h}U^{(k)}_{n+2k+1-h}(2889)$$

Theorem 2: let n be non-negative integer,L_n is the n^{th} Lucas numbers and$T_n(z)$is define by(1.1)then

(i)

$$\left(T_{2n-1}\left(\frac{5\sqrt{5}}{2}\right)- T_{2n-1}\left(\frac{-5\sqrt{5}}{2}\right)\right)^2 = \left(L_{5(2n-1)}\right)^2+4 \quad (1.10)$$

(ii)

$$T_{2n}\left(\frac{5\sqrt{5}}{2}\right)+ T_{2n}\left(\frac{-5\sqrt{5}}{2}\right) = L_{10(n)} \quad (1.11)$$

Proof: Equations (1.10) and(1.11) can be proved by taking$z = \pm\frac{5\sqrt{5}}{2}$, using equations (1.2), (1.6) and solving L.H.S of equations (1.10) and (1.11) to obtain R.H.S.

Theorem 3: let n be non-negative integer,L_n is the n^{th} Lucas numbers and $T_n(z)$ is the Chebyshev polynomial of first kind then

$$T_n\left(17\sqrt{5}\right)+ T_n\left(-17\sqrt{5}\right) = \frac{\left((-1)^n+1\right)}{2}L_{9(n)} \quad (1.12)$$

Proof: Equations (1.12) can be proved by taking $z = \pm 17\sqrt{5}$, using equations (1.2), (1.6) and solving L.H.S of equations (1.12) to obtain R.H.S.

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